Inequalities - Maximum and Minimum

1. Given $x_i > 0$ $(i = 1, 2, \ldots, n)$ and the sum $x_1 + x_2 + \ldots + x_n = C$, a constant.

Prove that the product $x_1x_2\ldots x_n$ reaches the greatest value when : $x_1 = x_2 = \ldots = x_n = \frac{C}{n}$.

2. Given $x_i > 0$ $(i = 1, 2, \ldots, n)$ and the product $x_1x_2\ldots x_n = C$, a constant.

Prove that the sum $x_1 + x_2 + \ldots + x_n$ attains the least value when: $x_1 = x_2 = \ldots = x_n = \sqrt[n]{C}$

3. Given $x_i > 0$ $(i = 1, 2, \ldots, n)$ and the sum $x_1 + x_2 + \ldots + x_n = C$, a constant.

Show that $x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$ takes on the greatest value when: $\frac{x_1}{\mu_1} = \frac{x_2}{\mu_2} = \ldots = \frac{x_n}{\mu_n} = \frac{C}{\mu_1 + \mu_2 + \ldots + \mu_n}$

where $\mu_i > 0$ $(i = 1, 2, \ldots, n)$ are rational.

4. Let $a_i > 0$, $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $a_1x_1 + a_2x_2 + \ldots + a_nx_n = C$.

Prove that the product $x_1x_2\ldots x_n$ reaches the greatest value when : $a_1x_1 + a_2x_2 + \ldots + a_nx_n = \frac{C}{n}$

5. Let $a_i > 0$, $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $a_1x_1^{\lambda_1} + a_2x_2^{\lambda_2} + \ldots + a_nx_n^{\lambda_n} = C$, where λ_i are rational.

Prove that $x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}$ takes on the greatest value when: $\frac{\lambda_1 a_1 x_1^{\lambda_1}}{\mu_1} = \frac{\lambda_2 a_2 x_2^{\lambda_2}}{\mu_2} = \ldots = \frac{\lambda_n a_n x_n^{\lambda_n}}{\mu_n}$

6. Let $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} = C$, a constant. Show that $a_1x_1^{\mu_1} + a_2x_2^{\mu_2} + \ldots + a_nx_n^{\mu_n}$ attains the least value

when: $\frac{x_1^{\mu_1}}{\frac{\lambda_1}{a_1\mu_1}} = \frac{x_2^{\mu_2}}{\frac{\lambda_2}{a_2\mu_2}} = \ldots = \frac{x_n^{\mu_n}}{\frac{\lambda_n}{a_n\mu_n}}$, where $a_i > 0$, $x_i > 0$; λ_i and $\mu_i > 0$ are rational.

7. Let $x + y + z = \frac{\pi}{2}$, $0 \le x, y, z \le \frac{\pi}{2}$.

At what values of x, y and z does the product $\tan x \tan y \tan z$ attain the greatest value?

8. Prove that $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} > 1$, where n is a positive integer.

9. If a, b, c, d, …. are p positive integers, whose sum is equal to n ,

show that the least value of $a!\, b!\, c!\, d! \ldots$ is $(q!)^{p-r}\, [(q + 1)!]^r$

where q is the quotient and r the remainder when n is divided by p .

10. Find the range of values of x for which $-2 < \frac{3x+11}{x+2} < 2$.

11. It is required to express $x^2 + 7y^2 + 20z^2 + 8yz - 2zx + 4xy$ in the form

$a(x + py + qz)^2 + b(y + rz)^2 + cz^2$, where a, b, c, q, r are constants.

By equating coefficients, or otherwise, determine the values of these constants.

Deduce that the given expression is never negative for real values of x, y, z .

12. If $x + y + z = 1$, $x, y, z \geq 0$,

show that the least value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is 9; and that $(1 - x)(1 - y)(1 - z) \geq 8xyz$.

13. Given that: if $a_1, a_2, \ldots, a_n$ are positive numbers, not all equal, then $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$,

prove that, if x, y and z are positive numbers such that $x + y + z = 1$, then x^2yz cannot be greater then $\frac{1}{64}$.

14. **(a)** Prove that $(a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2) \geq (a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)^2$,

where all the symbols denote real numbers.

(b) If $r^2x = aR + bR^3$, where $R \geq r$, and all the symbols are real and positive, prove that

$x \geq 2\sqrt{ab}$.